Differentiation

CHAPTER 3

Concepts and Skills
In this chapter, you will review

- derivatives as instantaneous rates of change;
- estimating derivatives using graphs and tables;
- derivatives of basic functions;
- the product, quotient, and chain rules;
- implicit differentiation;
- derivatives of inverse functions;
- Rolle's Theorem and the Mean Value Theorem.

In addition, BC Calculus students will review

- derivatives of parametrically defined functions;
- L'Hôpital's Rule for evaluating limits of indeterminate forms.

A. DEFINITION OF DERIVATIVE

Derivative

At any x in the domain of the function $y = f(x)$, the *derivative* is defined as

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} \quad \text{or} \quad \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}. \tag{1}$$

Differentiable

The function is said to be *differentiable* at every x for which this limit exists, and its derivative may be denoted by $f'(x)$, y', $\frac{dy}{dx}$, or $D_x y$. Frequently Δx is replaced by h or some other symbol.

The derivative of $y = f(x)$ at $x = a$, denoted by $f'(a)$ or $y'(a)$, may be defined as follows:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}. \tag{2}$$

Difference quotient

Average rate of change

Instantaneous rate of change

Slope of a curve

The fraction $\frac{f(a+h)-f(a)}{h}$ is called the *difference quotient for f at a* and represents the *average rate of change of f from a to a + h*. Geometrically, it is the slope of the secant PQ to the curve $y = f(x)$ through the points $P(a,f(a))$ and $Q(a+h,f(a+h))$. The limit, $f'(a)$, of the difference quotient is the (*instantaneous*) *rate of change of f* at point a. Geometrically, the derivative $f'(a)$ is the limit of the slope of secant PQ as Q approaches P; that is, as h approaches zero. This limit is the *slope of the curve at P*. The *tangent to the curve at P* is the line through P with this slope.

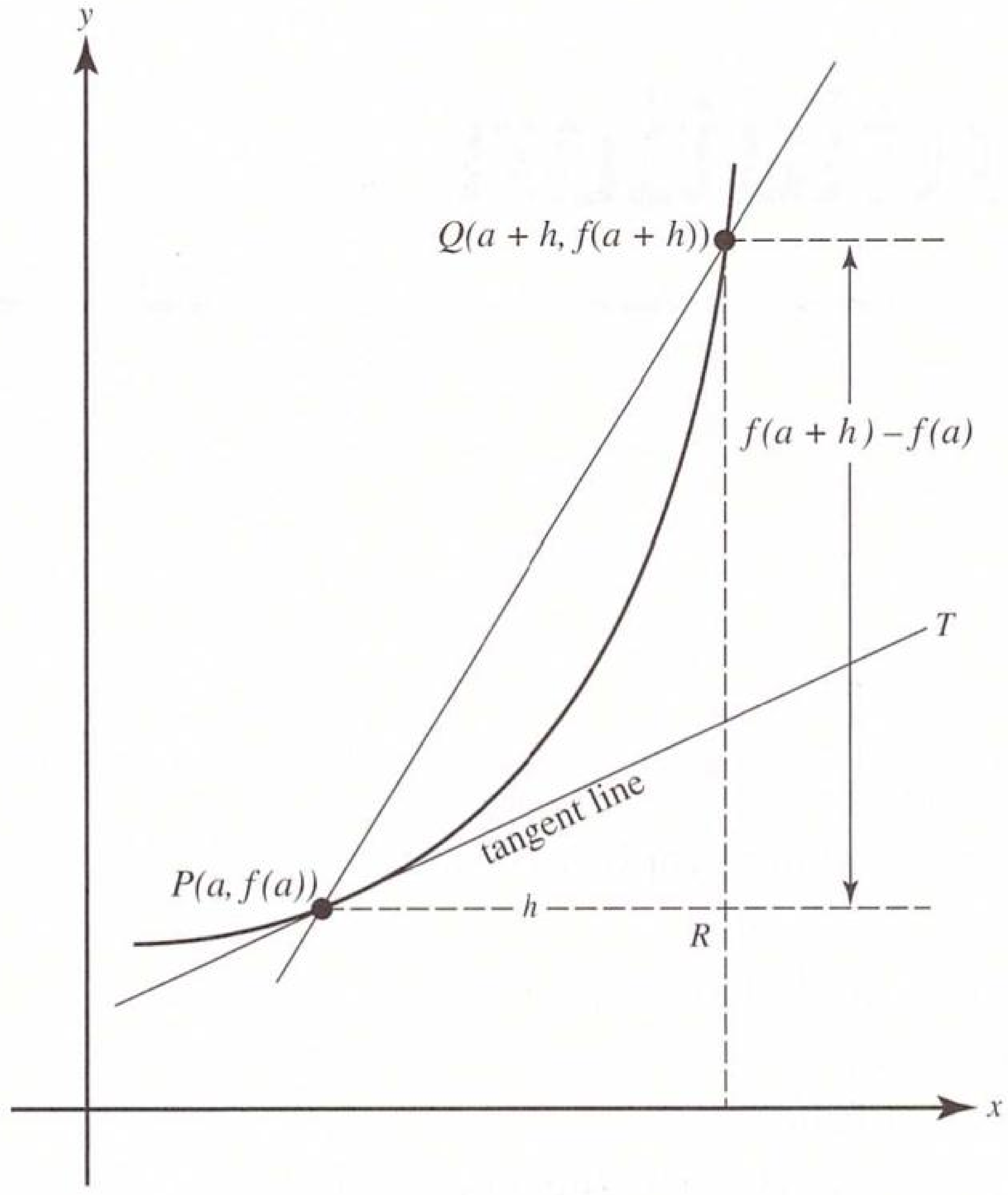

FIGURE N3–1a

In Figure N3–1a, PQ is the secant line through $(a,f(a))$ and $(a + h,f(a + h))$. The average rate of change from a to $a + h$ equals $\frac{RQ}{PR}$, which is the slope of secant PQ.

PT is the tangent to the curve at P. As h approaches zero, point Q approaches point P along the curve, PQ approaches PT, and the slope of PQ approaches the slope of PT, which equals $f'(a)$.

If we replace $(a + h)$ by x, in (2) above, so that $h = x - a$, we get the equivalent expression

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}. \tag{3}$$

See Figure N3–1b.

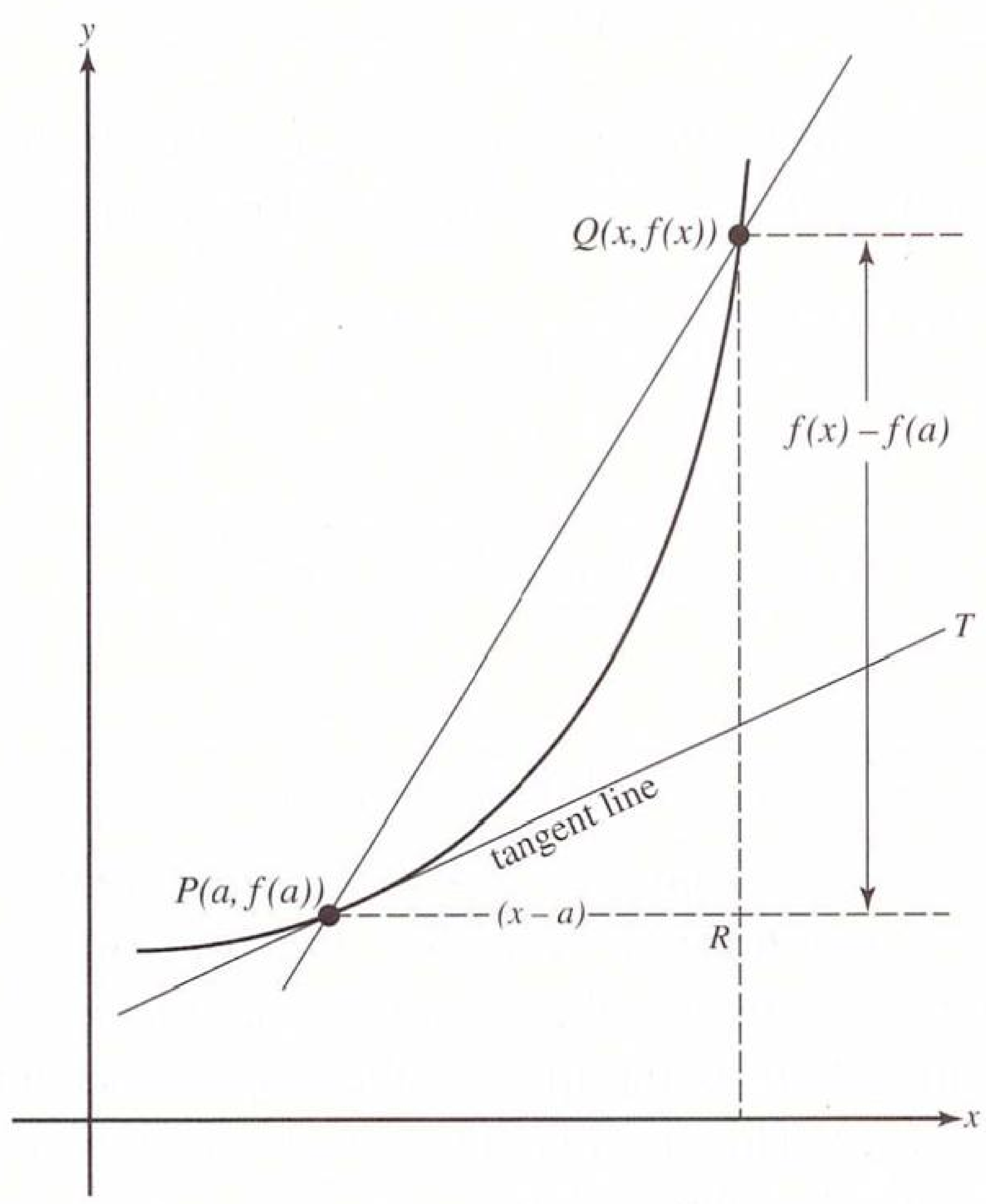

FIGURE N3–1b

The second derivative, denoted by $f''(x)$ or $\frac{d^2y}{dx^2}$ or y'', is the (first) derivative of $f'(x)$. Also, $f''(a)$ is the second derivative of $f(x)$ at $x = a$.

B. FORMULAS

The formulas in this section for finding derivatives are so important that familiarity with them is essential. If a and n are constants and u and v are differentiable functions of x, then:

$$\frac{da}{dx} = 0 \tag{1}$$

$$\frac{d}{dx}au = a\frac{du}{dx} \tag{2}$$

$$\frac{d}{dx}u^a = au^{a-1}\frac{du}{dx} \quad \text{(the Power Rule)}; \quad \frac{d}{dx}x^n = nx^{n-1} \tag{3}$$

$$\frac{d}{dx}(u+v) = \frac{d}{dx}u + \frac{d}{dx}v; \quad \frac{d}{dx}(u-v) = \frac{d}{dx}u - \frac{d}{dx}v \tag{4}$$

Product rule

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx} \quad \text{(the Product Rule)} \tag{5}$$

Quotient rule

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} \quad (v \neq 0) \quad \text{(the Quotient Rule)} \tag{6}$$

$$\frac{d}{dx}\sin u = \cos u\frac{du}{dx} \tag{7}$$

$$\frac{d}{dx}\cos u = -\sin u\frac{du}{dx} \tag{8}$$

$$\frac{d}{dx}\tan u = \sec^2 u\frac{du}{dx} \tag{9}$$

$$\frac{d}{dx}\cot u = -\csc^2 u\frac{du}{dx} \tag{10}$$

$$\frac{d}{dx}\sec u = \sec u \tan u\frac{du}{dx} \tag{11}$$

$$\frac{d}{dx}\csc u = -\csc u \cot u\frac{du}{dx} \tag{12}$$

$$\frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx} \tag{13}$$

$$\frac{d}{dx}e^u = e^u\frac{du}{dx} \tag{14}$$

$$\frac{d}{dx}a^u = a^u \ln a\frac{du}{dx} \tag{15}$$

$$\frac{d}{dx}\sin^{-1} u = \frac{d}{dx}\arcsin u = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx} \quad (-1 < u < 1) \tag{16}$$

$$\frac{d}{dx}\cos^{-1} u = \frac{d}{dx}\arccos u = -\frac{1}{\sqrt{1-u^2}}\frac{du}{dx} \quad (-1 < u < 1) \tag{17}$$

$$\frac{d}{dx}\tan^{-1}u = \frac{d}{dx}\text{ arctan } u = \frac{1}{1+u^2}\frac{du}{dx} \tag{18}$$

$$\frac{d}{dx}\cot^{-1}u = \frac{d}{dx}\text{ arccot } u = -\frac{1}{1+u^2}\frac{du}{dx} \tag{19}$$

$$\frac{d}{dx}\sec^{-1}u = \frac{d}{dx}\text{ arcsec } u = \frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx} \quad (|u|>1) \tag{20}$$

$$\frac{d}{dx}\csc^{-1}u = \frac{d}{dx}\text{ arccsc } u = -\frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx} \quad (|u|>1) \tag{21}$$

C. THE CHAIN RULE; THE DERIVATIVE OF A COMPOSITE FUNCTION

Formula (3) on page 113 says that

$$\boxed{\frac{d}{dx}u^a = au^{a-1}\frac{du}{dx}.}$$

This formula is an application of the *Chain Rule*. For example, if we use formula (3) to find the derivative of $(x^2 - x + 2)^4$, we get

$$\frac{d}{dx}(x^2-x+2)^4 = 4(x^2-x+2)^3\cdot(2x-1).$$

In this last equation, if we let $y = (x^2 - x + 2)^4$ and let $u = x^2 - x + 2$, then $y = u^4$. The preceding derivative now suggests one form of the Chain Rule:

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = 4u^3\cdot\frac{du}{dx} = 4(x^2-x+2)^3\cdot(2x-1)$$

as before. Formula (3) on page 113 gives the general case where $y = u^n$ and u is a differentiable function of x.

Now suppose we think of y as the composite function $f(g(x))$, where $y = f(u)$ and $u = g(x)$ are differentiable functions. Then

Chain rule

$$\begin{aligned}(f(g(x)))' &= f'(g(x))\cdot g'(x)\\ &= f'(u)\cdot g'(x)\\ &= \frac{dy}{du}\cdot\frac{du}{dx}.\end{aligned}$$

as we obtained above. The Chain Rule tells us how to differentiate the composite function: "Find the derivative of the 'outside' function first, then multiply by the derivative of the 'inside' one."

For example:

$$\frac{d}{dx}(x^3+1)^{10} = 10(x^3+1)^9\cdot 3x^2 = 30x^2(x^3+1)^9,$$

$$\frac{d}{dx}\sqrt{7x-2} = \frac{d}{dx}(7x-2)^{1/2} = \frac{1}{2}(7x-2)^{-1/2}\cdot 7,$$

$$\frac{d}{dx}\frac{3}{(2-4x^2)^4} = \frac{d}{dx}3(2-4x^2)^{-4} = 3\cdot(-4)(2-4x^2)^{-5}\cdot(-8x),$$

$$\frac{d}{dx}\sin\left(\frac{\pi}{2}-x\right)=\cos\left(\frac{\pi}{2}-x\right)\cdot(-1),$$

$$\frac{d}{dx}\cos^3 2x=\frac{d}{dx}(\cos 2x)^3=3(\cos 2x)^2\cdot(-\sin 2x\cdot 2).$$

Many of the formulas listed above in §B and most of the illustrative examples that follow use the Chain Rule. Often the chain rule is used more than once in finding a derivative.

Note that the algebraic simplifications that follow are included only for completeness.

EXAMPLE 1

If $y=4x^3-5x+7$, find $y'(1)$ and $y''(1)$.

SOLUTION: $y'=\dfrac{dy}{dx}=12x^2-5$ and $y''=\dfrac{d^2y}{dx^2}=24x.$

Then $y'(1)=12\cdot 1^2-5=7$ and $y''(1)=24\cdot 1=24$.

EXAMPLE 2

If $f(x)=(3x+2)^5$, find $f'(x)$.

SOLUTION: $f'(x)=5(3x+2)^4\cdot 3=15(3x+2)^4$.

EXAMPLE 3

If $y=\sqrt{3-x-x^2}$, find $\dfrac{dy}{dx}$.

SOLUTION: $y=(3-x-x^2)^{1/2}$ so, $\dfrac{dy}{dx}=\dfrac{1}{2}(3-x-x^2)^{-1/2}(-1-2x)$

$$=-\frac{1+2x}{2\sqrt{3-x-x^2}}.$$

EXAMPLE 4

If $y=\dfrac{5}{\sqrt{(1-x^2)^3}}$, find $\dfrac{dy}{dx}$.

SOLUTION: $y=5(1-x^2)^{-3/2}$ so $\dfrac{dy}{dx}=\dfrac{-15}{2}(1-x^2)^{-5/2}(-2x)$

$$=\frac{15x}{(1-x^2)^{5/2}}.$$

EXAMPLE 5

If $s(t)=(t^2+1)(1-t)^2$, find $s'(t)$.

SOLUTION: $s'(t)=(t^2+1)\cdot 2(1-t)(-1)+(1-t)^2\cdot 2t$ (Product Rule)

$=2(1-t)(-1+t-2t^2)$.

EXAMPLE 6

If $f(t)=e^{2t}\sin 3t$, find $f'(0)$.

SOLUTION: $f'(t)=e^{2t}(\cos 3t\cdot 3)+\sin 3t(e^{2t}\cdot 2)$ (Product Rule)

$=e^{2t}(3\cos 3t+2\sin 3t)$

Then, $f'(0)=1(3\cdot 1+2\cdot 0)=3$.

EXAMPLE 7

If $f(v) = \frac{2v}{1-2v^2}$, find $f'(v)$.

SOLUTION: $f'(v) = \frac{(1-2v^2)\cdot 2 - 2v(-4v)}{(1-2v^2)^2} = \frac{2+4v^2}{(1-2v^2)^2}$. (Quotient Rule)

Note that neither $f(v)$ nor $f'(v)$ exists where the denominator equals zero, namely, where $1 - 2v^2 = 0$ or where v equals $\pm\frac{\sqrt{2}}{2}$.

EXAMPLE 8

If $f(x) = \frac{\sin x}{x^2}$, $x \neq 0$, find $f'(x)$.

SOLUTION: $f'(x) = \frac{x^2 \cos x - \sin x \cdot 2x}{x^4} = \frac{x\cos x - 2\sin x}{x^3}$.

EXAMPLE 9

If $y = \tan(2x^2 + 1)$, find y'.

SOLUTION: $y' = 4x\sec^2(2x^2 + 1)$.

EXAMPLE 10

If $x = \cos^3(1 - 3\theta)$, find $\frac{dx}{d\theta}$.

SOLUTION:

$$\frac{dx}{d\theta} = -3\cos^2(1-3\theta)\sin(1-3\theta)(-3)$$

$$= 9\cos^2(1-3\theta)\sin(1-3\theta).$$

EXAMPLE 11

If $y = e^{(\sin x)+1}$, find $\frac{dy}{dx}$.

SOLUTION: $\frac{dy}{dx} = \cos x \cdot e^{(\sin x)+1}$.

EXAMPLE 12

If $y = (x+1)\ln^2(x+1)$, find $\frac{dy}{dx}$.

SOLUTION: $\frac{dy}{dx} = (x+1)\frac{2\ln(x+1)}{x+1} + \ln^2(x+1)$ (Product and Chain Rules)

$$= 2\ln(x+1) + \ln^2(x+1).$$

EXAMPLE 13

If $g(x) = (1 + \sin^2 3x)^4$, find $g'\left(\frac{\pi}{2}\right)$.

SOLUTION: $g'(x) = 4\ (1 + \sin^2 3x)^3(2 \sin 3x \cos 3x) \cdot (3)$

$= 24\ (1 + \sin^2 3x)^3(\sin 3x \cos 3x)$.

Then $g'\left(\frac{\pi}{2}\right) = 24(1 + (-1)^2)^3(-1 \cdot 0) = 24 \cdot 8 \cdot 0 = 0$.

EXAMPLE 14

If $y = \sin^{-1} x + x\sqrt{1-x^2}$, find y'.

SOLUTION:
$$y' = \frac{1}{\sqrt{1-x^2}} + \frac{x(-2x)}{2\sqrt{1-x^2}} + \sqrt{1-x^2}$$
$$= \frac{1-x^2+1-x^2}{\sqrt{1-x^2}} = 2\sqrt{1-x^2}.$$

EXAMPLE 15

If $u = \ln \sqrt{v^2 + 2v - 1}$, find $\frac{du}{dv}$.

SOLUTION: $u = \frac{1}{2} \ln (v^2 + 2v - 1)$ so
$$\frac{du}{dv} = \frac{1}{2}\frac{2v+2}{v^2+2v-1} = \frac{v+1}{v^2+2v-1}.$$

EXAMPLE 16

If $s = e^{-t}(\sin t - \cos t)$, find s'.

SOLUTION:
$$s' = e^{-t}(\cos t + \sin t) + (\sin t - \cos t)(-e^{-t})$$
$$= e^{-t}(2\cos t) = 2e^{-t} \cos t.$$

EXAMPLE 17

Let $y = 2u^3 - 4u^2 + 5u - 3$ and $u = x^2 - x$. Find $\frac{dy}{dx}$.

SOLUTION: $\frac{dy}{dx} = (6u^2 - 8u + 5)(2x - 1)$
$$= [6(x^2 - x)^2 - 8(x^2 - x) + 5](2x - 1).$$

EXAMPLE 18

If $y = \sin (ax + b)$, with a and b constants, find $\frac{dy}{dx}$.

SOLUTION: $\frac{dy}{dx} = [\cos(ax + b)] \cdot a = a\cos(ax + b)$.

EXAMPLE 19

If $f(x) = ae^{kx}$ (with a and k constants), find f' and f''.

SOLUTION: $f'(x) = kae^{kx}$ and $f'' = k^2ae^{kx}$.

EXAMPLE 20

If $y = \ln (kx)$, where k is a constant, find $\frac{dy}{dx}$.

SOLUTION: We can use both formula (13), page 113, and the Chain Rule to get

$$\frac{dy}{dx} = \frac{1}{kx} \cdot k = \frac{1}{x}.$$

Alternatively, we can rewrite the given function using a property of logarithms: $\ln (kx) = \ln k + \ln x$. Then

$$\frac{dy}{dx} = 0 + \frac{1}{x} = \frac{1}{x}.$$

EXAMPLE 21

Given $f(u) = u^2 - u$ and $u = g(x) = x^3 - 5$ and $F(x) = f(g(x))$, evaluate $F'(2)$.

SOLUTION: $F'(2) = f'(g(2))g'(2) = f'(3) \cdot (12) = 5 \cdot 12 = 60$.

Now, since $g'(x) = 3x^2$, $g'(2) = 12$, and since $f'(u) = 2u - 1$, $f'(3) = 5$.
Of course, we get exactly the same answer as follows.
Since $F(x) = (x^3 - 5)^2 - (x^3 - 5)$,

$$F'(x) = 2(x^3 - 5) \cdot 3x^2 - 3x^2,$$

$$F'(2) = 2 \cdot (3) \cdot 12 - 12 = 60.$$

D. DIFFERENTIABILITY AND CONTINUITY

If a function f has a derivative at $x = c$, then f is continuous at $x = c$.

This statement is an immediate consequence of the definition of the derivative of $f'(c)$ in the form

$$\boxed{f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}.}$$

If $f'(c)$ exists, then it follows that $\lim_{x \to c} f(x) = f(c)$, which guarantees that f is continuous at $x = c$.

If f is differentiable at c, its graph cannot have a hole or jump at c, nor can $x = c$ be a vertical asymptote of the graph. The tangent to the graph of f cannot be vertical at $x = c$; there cannot be a corner or cusp at $x = c$.

Each of the "prohibitions" in the preceding paragraph (each "cannot") tells how a function may fail to have a derivative at c. These cases are illustrated in Figures N3–2 (a) through (f).

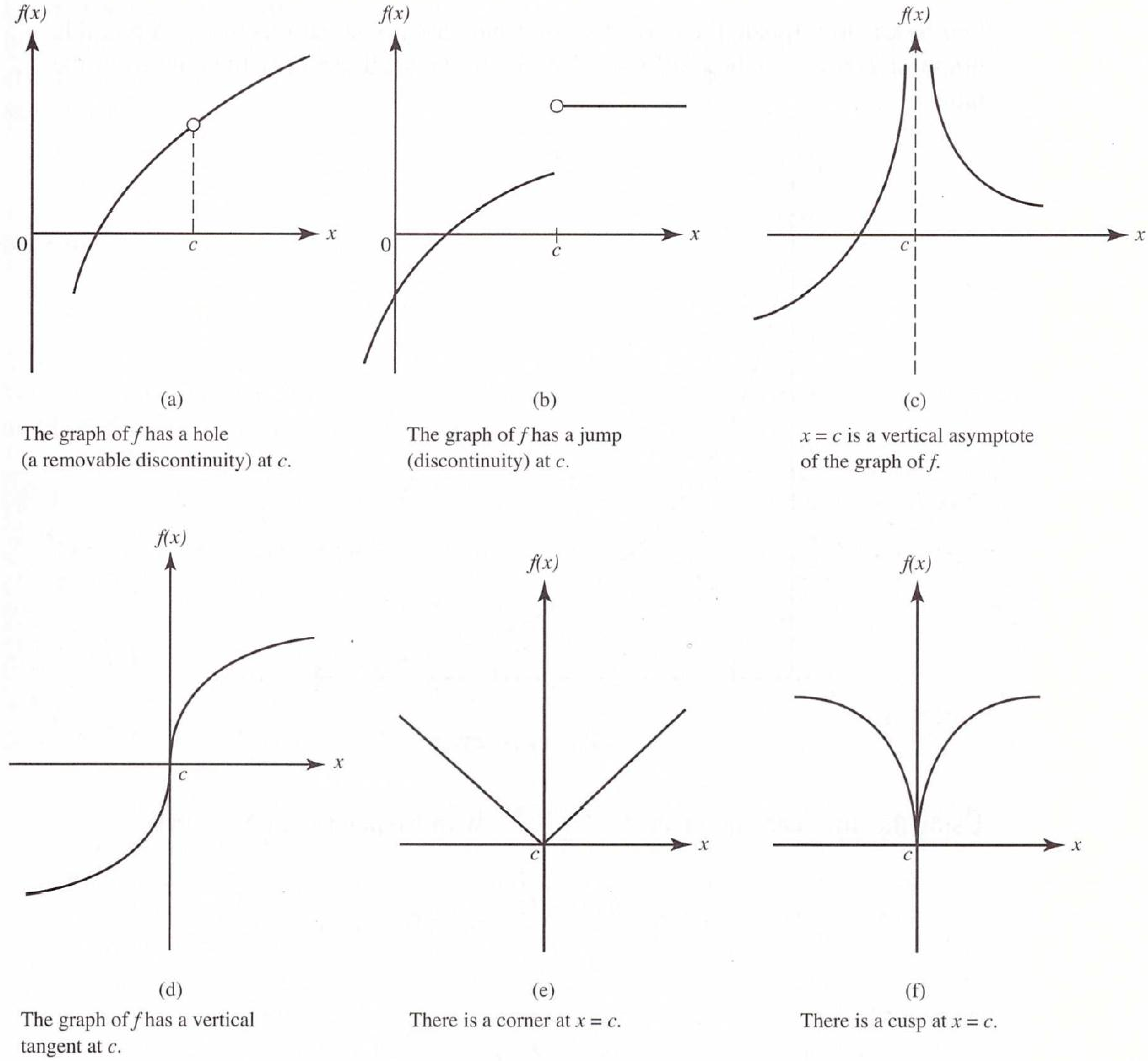

FIGURE N3–2

The graph in (e) is for the absolute function, $f(x) = |x|$. Since $f'(x) = -1$ for all negative x but $f'(x) = +1$ for all positive x, $f'(0)$ does not exist.

We may conclude from the preceding discussion that, although differentiability implies continuity, the converse is false. The functions in (d), (e), and (f) in Figure N3–2 are all continuous at $x = 0$, but not one of them is differentiable at the origin.

E. ESTIMATING A DERIVATIVE

E1. Numerically.

EXAMPLE 22

The table shown gives the temperatures of a polar bear on a very cold arctic day (t = minutes; T = degrees Fahrenheit):

t	0	1	2	3	4	5	6	7	8
T	98	94.95	93.06	91.90	91.17	90.73	90.45	90.28	90.17

Our task is to estimate the derivative of T numerically at various times. A possible graph of $T(t)$ is sketched in Figure N3–3, but we shall use only the data from the table.

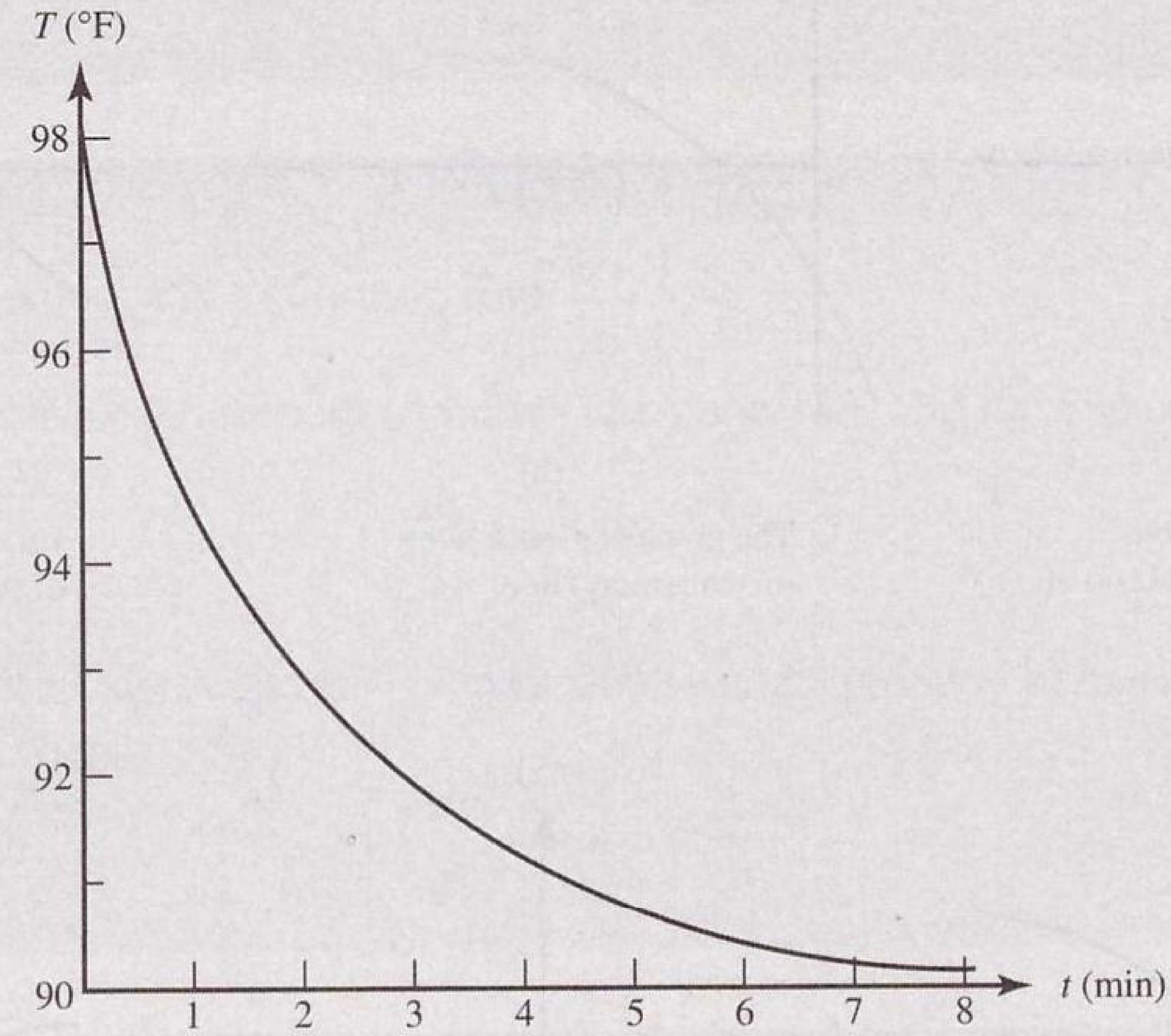

FIGURE N3–3

Using the difference quotient $\frac{T(t+h)-T(t)}{h}$ with h equal to 1, we see that

$$T'(0) \simeq \frac{T(1)-T(0)}{1} = -3.05°/\text{min}.$$

Also,

$$T'(1) \simeq \frac{T(2)-T(1)}{1} = -1.89°/\text{min},$$

$$T'(2) \simeq \frac{T(3)-T(2)}{1} = -1.16°/\text{min},$$

$$T'(3) \simeq \frac{T(4)-T(3)}{1} = -0.73°/\text{min},$$

and so on.

The following table shows the *approximate* values of $T'(t)$ obtained from the difference quotients above:

t	0	1	2	3	4	5	6	7
$T'(t)$	–3.05	–1.89	–1.16	–0.73	–0.47	–0.28	–0.17	–0.11

Note that the entries for $T'(t)$ also represent the approximate slopes of the T curve at times 0.5, 1.5, 2.5, and so on.

From a Symmetric Difference Quotient

In Example 22 we approximated a derivative numerically from a table of values. We can also estimate $f'(a)$ numerically using the *symmetric difference quotient,* which is defined as follows:

$$\boxed{f'(a) \simeq \frac{f(a+h)-f(a-h)}{2h}.}$$

Symmetric difference quotient

Note that the symmetric difference quotient is equal to

$$\boxed{\frac{1}{2}\left[\frac{f(a+h)-f(a)}{h}+\frac{f(a)-f(a-h)}{h}\right].}$$

We see that it is just the average of two difference quotients. Many calculators use the symmetric difference quotient in finding derivatives.

EXAMPLE 23

For the function $f(x) = x^4$, approximate $f'(1)$ using the symmetric difference quotient with $h = 0.01$.

SOLUTION: $$f'(1) \simeq \frac{(1.01)^4 - (0.99)^4}{2(0.01)} = 4.0004.$$

The exact value of $f'(1)$, of course, is 4.

The use of the symmetric difference quotient is particularly convenient when, as is often the case, obtaining a derivative precisely (with formulas) is cumbersome and an approximation is all that is needed for practical purposes.

A word of caution is in order. Sometimes a wrong result is obtained using the symmetric difference quotient. On pages 118 and 119 we noted that $f(x) = |x|$ does not have a derivative at $x = 0$, since $f'(x) = -1$ for all $x < 0$ but $f'(x) = 1$ for all $x > 0$. Our calculator (which uses the symmetric difference quotient) tells us (incorrectly!) that $f'(0) = 0$. Note that, if $f(x) = |x|$, the symmetric difference quotient gives 0 for $f'(0)$ for every $h \neq 0$. If, for example, $h = 0.01$, then we get

$$f'(0) \simeq \frac{|0.01| - |-0.01|}{0.02} = \frac{0}{0.02} = 0,$$

which, as previously noted, is incorrect. The graph of the derivative of $f(x) = |x|$, which we see in Figure N3–4, shows that $f'(0)$ does not exist.

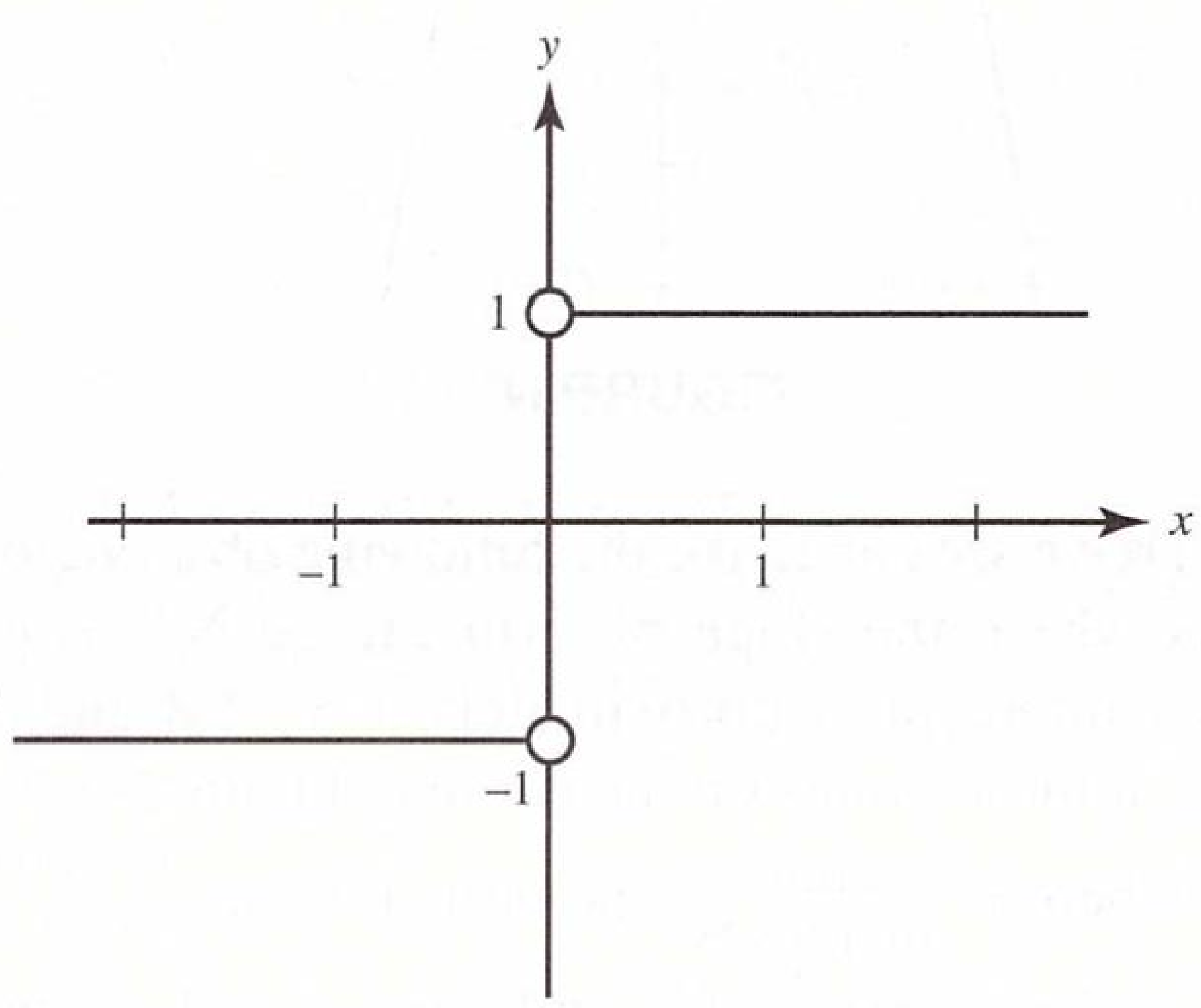

FIGURE N3–4

E2. Graphically.

If we have the graph of a function $f(x)$, we can use it to graph $f'(x)$. We accomplish this by estimating the slope of the graph of $f(x)$ at enough points to assure a smooth curve for $f'(x)$. In Figure N3–5 we see the graph of $y = f(x)$. Below it is a table of the approximate slopes estimated from the graph.

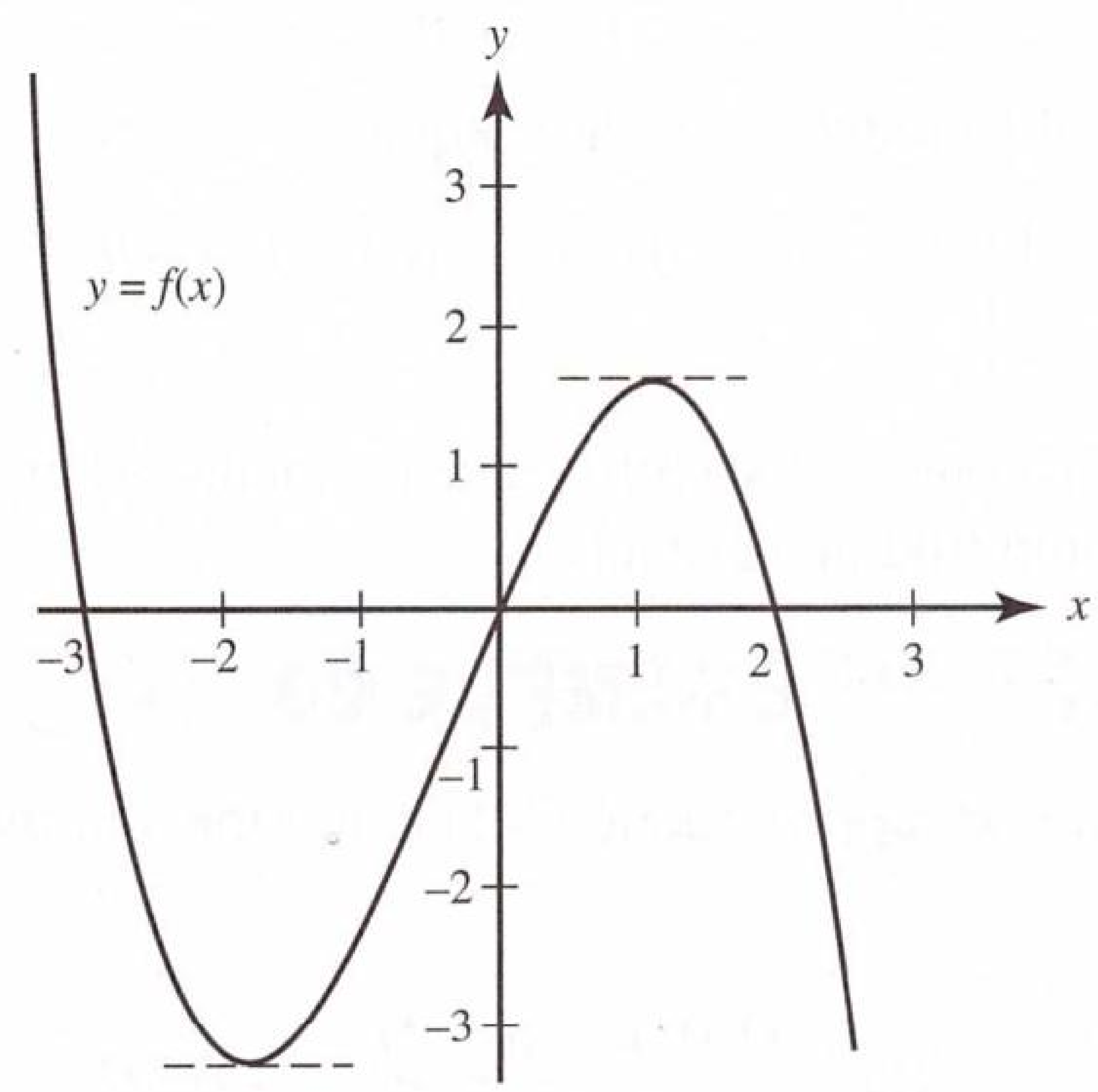

FIGURE N3–5

x	–3	–2.5	–2	–1.5	–1	0	0.5	1	1.5	2	2.5
$f'(x)$	–6	–3	–0.5	1	2	2	1.5	0.5	–2	–4	–7

Figure N3–6 was obtained by plotting the points from the table of slopes above and drawing a smooth curve through these points. The result is the graph of $y = f'(x)$.

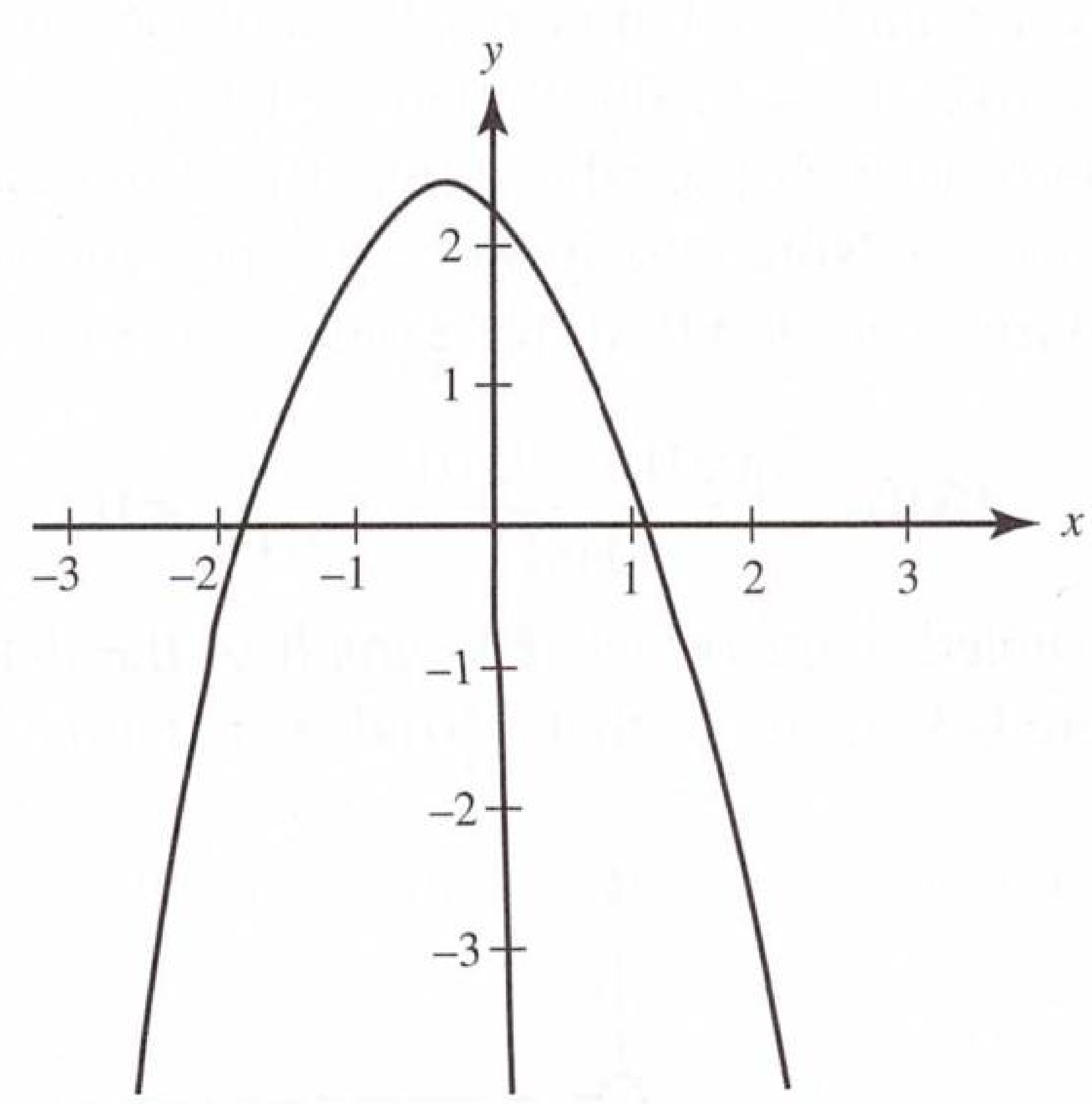

FIGURE N3–6

From the graphs above we can make the following observations:

(1) At the points where the slope of f (in Figure N3–5) equals 0, the graph of f' (Figure N3–6) has x-intercepts: approximately $x = -1.8$ and $x = 1.1$. We've drawn horizontal broken lines at these points on the curve in Figure N3-5.

(2) On intervals where f $\genfrac{}{}{0pt}{}{\text{decreases}}{\text{increases}}$, the derivative is $\genfrac{}{}{0pt}{}{\text{negative}}{\text{positive}}$. We see here that f decreases for $x < -1.8$ (approximately) and for $x > 1.1$ (approximately), and that f increases for $-1.8 < x < 1.1$ (approximately). In Chapter 4 we discuss other behaviors of f that are reflected in the graph of f'.

BC ONLY

Parametric differentiation

F. DERIVATIVES OF PARAMETRICALLY DEFINED FUNCTIONS

Parametric equations were defined on page 77.

If $x = f(t)$ and $y = g(t)$ are differentiable functions of t, then

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \quad \text{and} \quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}.$$

EXAMPLE 24

If $x = 2\sin\theta$ and $y = \cos 2\theta$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

SOLUTION: $\dfrac{dy}{dx} = \dfrac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \dfrac{-2\sin 2\theta}{2\cos\theta} = -\dfrac{2\sin\theta\cos\theta}{\cos\theta} = -2\sin\theta.$

Also,

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{d\theta}\left(\frac{dy}{dx}\right)}{\frac{dx}{d\theta}} = \frac{-2\cos\theta}{2\cos\theta} = -1.$$

EXAMPLE 25

Find the equation of the tangent to the curve in Example 24 for $\theta = \frac{\pi}{6}$.

SOLUTION:
When $\theta = \frac{\pi}{6}$, the slope of the tangent, $\frac{dy}{dx}$, equals $-2\sin\left(\frac{\pi}{6}\right) = -1$. Since $x = 2\sin\left(\frac{\pi}{6}\right) = 1$ and $y = \cos\left(2 \cdot \frac{\pi}{6}\right) = \cos\frac{\pi}{3} = \frac{1}{2}$, the equation is

$$y - \frac{1}{2} = -1(x-1) \quad \text{or} \quad y = -x + \frac{3}{2}.$$

EXAMPLE 26

Suppose two objects are moving in a plane during the time interval $0 \leqslant t \leqslant 4$. Their positions at time t are described by the parametric equations

$$x_1 = 2t, \qquad y_1 = 4t - t^2 \qquad \text{and} \qquad x_2 = t + 1, \qquad y_2 = 4 - t.$$

(a) Find all collision points. Justify your answer.
(b) Use a calculator to help you sketch the paths of the objects, indicating the direction in which each object travels.

BC ONLY

SOLUTION:

(a) Equating x_1 and x_2 yields $t = 1$. When $t = 1$, both y_1 and y_2 equal 3. So $t = 1$ yields a *true* collision point (not just an intersection point) at (2,3). (An *intersection point* is any point that is on both curves, but not necessarily at the same time.)

(b) Using parametric mode, we graph both curves with t in [0,4], in the window $[0,8] \times [0,4]$ as shown in Figure N3–7.

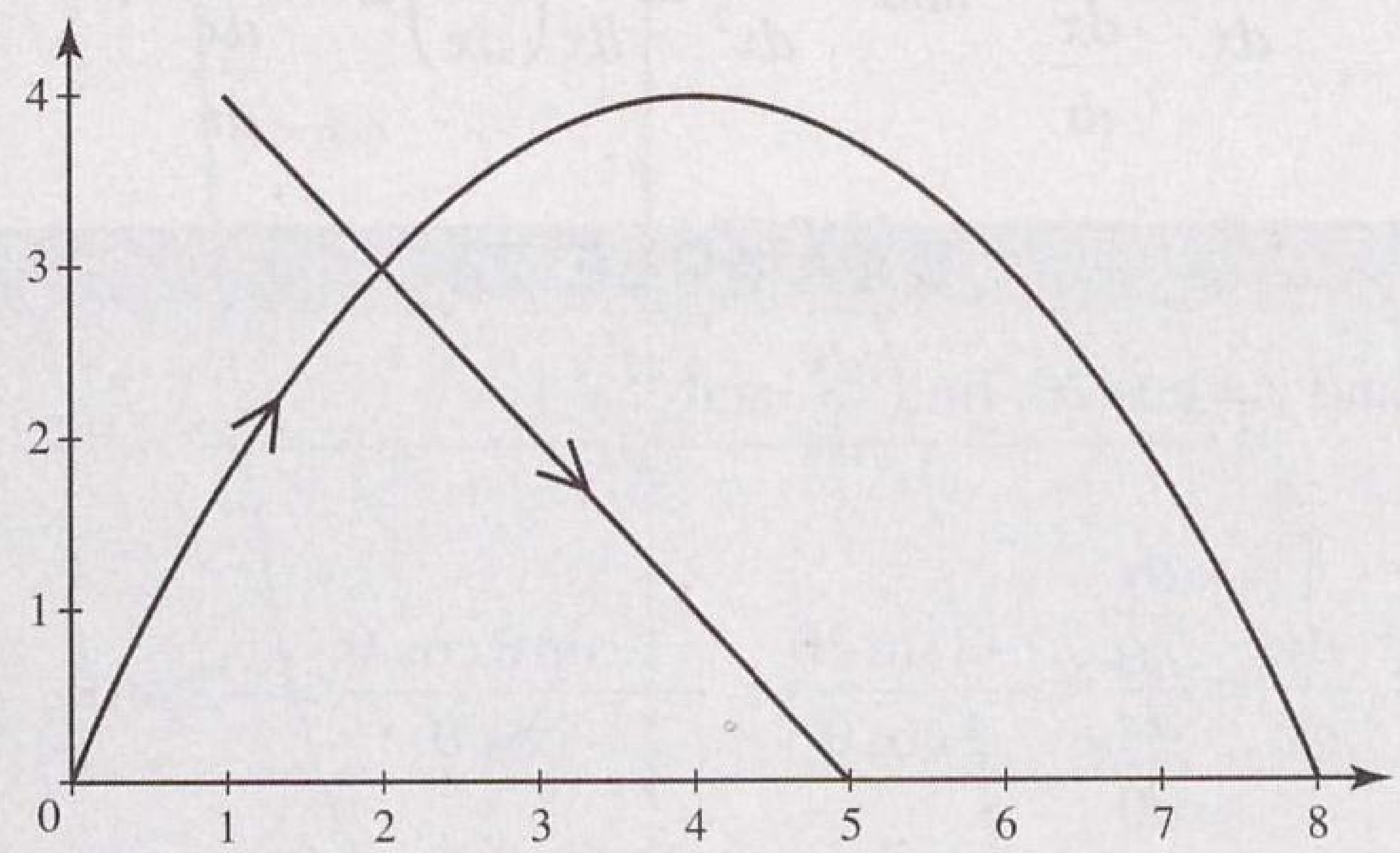

FIGURE N3–7

We've inserted arrows to indicate the direction of motion.

Note that if we draw the curves in simultaneous graphing mode, we can watch the objects as they move, seeing that they do indeed pass through the intersection point at the same time.

G. IMPLICIT DIFFERENTIATION

When a functional relationship between x and y is defined by an equation of the form $F(x,y) = 0$, we say that the equation defines y *implicitly* as a function of x. Some examples are $x^2 + y^2 - 9 = 0$, $y^2 - 4x = 0$, and $\cos(xy) = y^2 - 5$ (which can be written as $\cos(xy) - y^2 + 5 = 0$). Sometimes two (or more) explicit functions are defined by $F(x,y) = 0$. For example, $x^2 + y^2 - 9 = 0$ defines the two functions $y_1 = +\sqrt{9 - x^2}$ and $y_2 = -\sqrt{9 - x^2}$, the upper and lower halves, respectively, of the circle centered at the origin with radius 3. Each function is differentiable except at the points where $x = 3$ and $x = -3$.

Implicit differentiation is the technique we use to find a derivative when y is not defined explicitly in terms of x but is differentiable.

In the following examples, we differentiate both sides with respect to x, using appropriate formulas, and then solve for $\frac{dy}{dx}$.

EXAMPLE 27

If $x^2 + y^2 - 9 = 0$, then

$$2x + 2y\frac{dy}{dx} = 0 \quad \text{and} \quad \frac{dy}{dx} = -\frac{x}{y}.$$

Note that the derivative above holds for every point on the circle, and exists for all y different from 0 (where the tangents to the circle are vertical).

EXAMPLE 28

If $x^2 - 2xy + 3y^2 = 2$, find $\frac{dy}{dx}$.

SOLUTION: $2x - 2\left(x\frac{dy}{dx} + y \cdot 1\right) + 6y\frac{dy}{dx} = 0$

$$\frac{dy}{dx}(6y - 2x) = 2y - 2x, \text{ so } \frac{dy}{dx} = \frac{y - x}{3y - x}.$$

EXAMPLE 29

If $x \sin y = \cos(x + y)$, find $\frac{dy}{dx}$.

SOLUTION: $x \cos y\frac{dy}{dx} + \sin y = -\sin(x + y)\left(1 + \frac{dy}{dx}\right)$,

$$\frac{dy}{dx} = -\frac{\sin y + \sin(x + y)}{x \cos y + \sin(x + y)}.$$

EXAMPLE 30

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ using implicit differentiation on the equation $x^2 + y^2 = 1$.

SOLUTION: $2x + 2y\frac{dy}{dx} = 0 \quad \rightarrow \quad \frac{dy}{dx} = -\frac{x}{y}$. (1)

Then

$$\frac{d^2y}{dx^2} = -\frac{y \cdot 1 - x\left(\frac{dy}{dx}\right)}{y^2} = -\frac{y - x\left(-\frac{x}{y}\right)}{y^2} \qquad (2)$$

$$= -\frac{y^2 + x^2}{y^3} = -\frac{1}{y^3}, \qquad (3)$$

where we substituted for $\frac{dy}{dx}$ from (1) in (2), then used the given equation to simplify in (3).

EXAMPLE 31

Using implicit differentiation, verify the formula for the derivative of the inverse sine function, $y = \sin^{-1} x = \arcsin x$, with domain $[-1,1]$ and range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

SOLUTION: $y = \sin^{-1} x \quad \leftrightarrow \quad x = \sin y$.

Now we differentiate with respect to x:

$$1 = \cos y \frac{dy}{dx},$$

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{+\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}},$$

where we chose the positive sign for cos y since cos y is nonnegative if $-\frac{\pi}{2} < y < \frac{\pi}{2}$. Note that this derivative exists only if $-1 < x < 1$.

H. DERIVATIVE OF THE INVERSE OF A FUNCTION

Suppose f and g are inverse functions. What is the relationship between their derivatives? Recall that the graphs of inverse functions are the reflections of each other in the line $y = x$, and that at corresponding points their x- and y-coordinates are interchanged.

Figure N3–8 shows a function f passing through point (a,b) and the line tangent to f at that point. The slope of the curve there, $f'(a)$, is represented by the ratio of the legs of the triangle, $\frac{dy}{dx}$. When this figure is reflected across the line $y = x$, we obtain the graph of f^{-1}, passing through point (b,a), with the horizontal and vertical sides of the slope

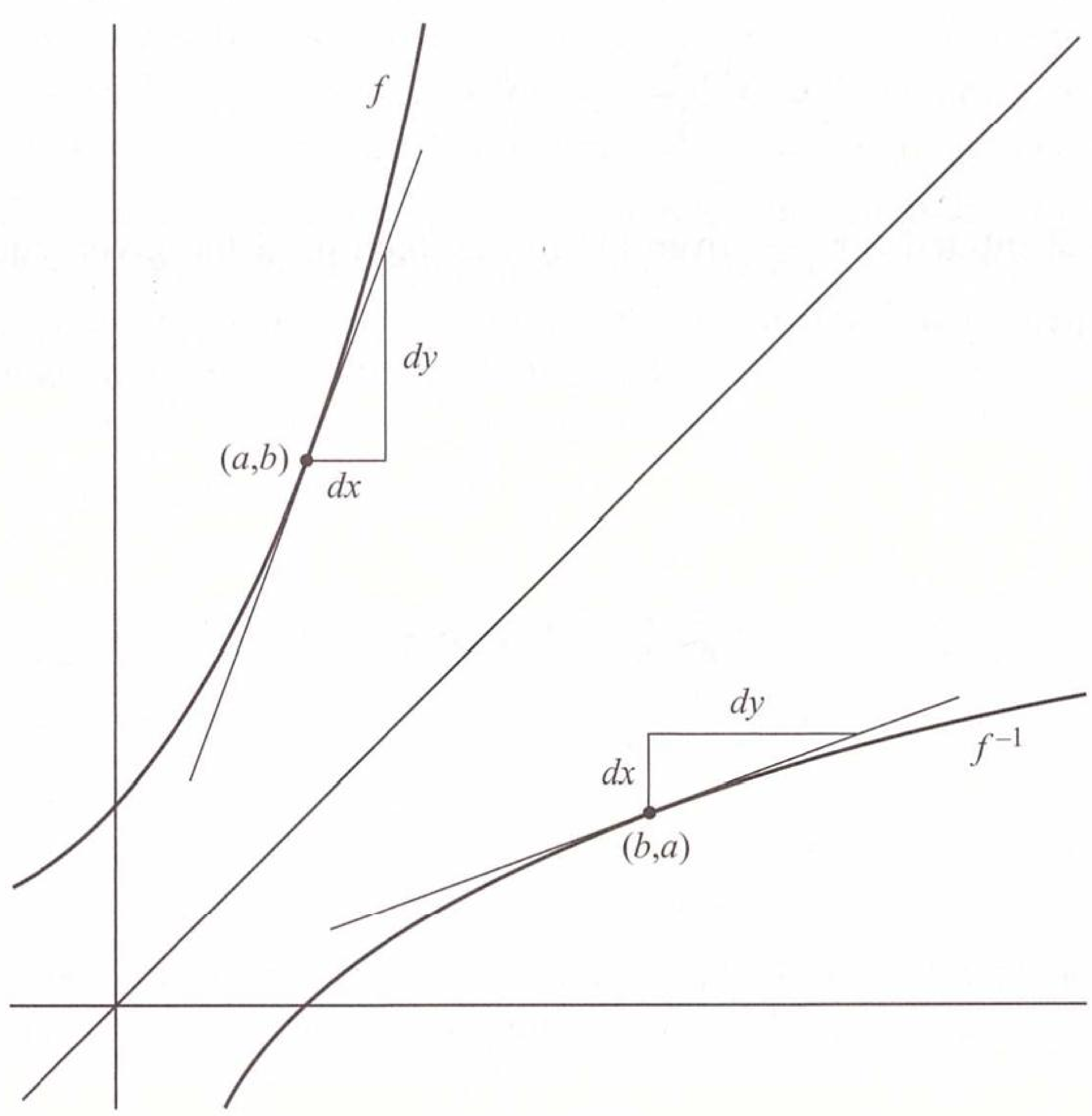

FIGURE N3–8

triangle interchanged. Note that the slope of the line tangent to the graph of f^{-1} at $x = b$ is represented by $\frac{dx}{dy}$, the reciprocal of the slope of f at $x = a$. We have, therefore,

Derivative of inverse function

$$\left(f^{-1}\right)'(b) = \frac{1}{f'(a)} \quad \text{or} \quad \left(f^{-1}\right)'(x) = \frac{1}{f'\left(f^{-1}(x)\right)}.$$

Simply put, the derivative of the inverse of a function at a point is the *reciprocal* of the derivative of the function *at the corresponding point.*

EXAMPLE 32

If $f(3) = 8$ and $f'(3) = 5$, what do we know about f^{-1}?

SOLUTION: Since f passes through the point (3,8), f^{-1} must pass through the point (8,3). Furthermore, since the graph of f has slope 5 at (3,8), the graph of f^{-1} must have slope $\frac{1}{5}$ at (8,3).

EXAMPLE 33

A function f and its derivative take on the values shown in the table. If g is the inverse of f, find $g'(6)$.

x	$f(x)$	$f'(x)$
2	6	$\frac{1}{3}$
6	8	$\frac{3}{2}$

SOLUTION: To find the slope of g at the point where $x = 6$, we must look at the point on f where $y = 6$, namely, (2,6). Since $f'(2) = \frac{1}{3}$, $g'(6) = 3$.

EXAMPLE 34

Let $y = f(x) = x^3 + x - 2$, and let g be the inverse function. Evaluate $g'(0)$.

SOLUTION: Since $f'(x) = 3x^2 + 1$, $g'(y) = \frac{1}{3x^2+1}$. To find x when $y = 0$, we must solve the equation $x^3 + x - 2 = 0$. Note by inspection that $x = 1$, so

$$g'(0) = \frac{1}{3(1)^2+1} = \frac{1}{4}.$$

EXAMPLE 35

Where is the tangent to the curve $4x^2 + 9y^2 = 36$ vertical?

SOLUTION: We differentiate the equation implicitly to get $\frac{dy}{dx}$: $8x + 18y\frac{dy}{dx} = 0$, so $\frac{dy}{dx} = -\frac{4x}{9y}$. Since the tangent line to a curve is vertical when $\frac{dx}{dy} = 0$, we conclude that $-\frac{9y}{4x}$ must equal zero; that is, y must equal zero. When we substitute $y = 0$ in the original equation, we get $x = \pm 3$. The points $(\pm 3,0)$ are the ends of the major axis of the ellipse, where the tangents are indeed vertical.

I. THE MEAN VALUE THEOREM

Mean Value Theorem

If the function $f(x)$ is continuous at each point on the closed interval $a \leqslant x \leqslant b$ and has a derivative at each point on the open interval $a < x < b$, then there is at least one number c, $a < c < b$, such that $\frac{f(b)-f(a)}{b-a} = f'(c)$. This important theorem, which relates average rate of change and instantaneous rate of change, is illustrated in Figure N3–9. For the function sketched in the figure there are two numbers, c_1 and c_2, between a and b where the slope of the curve equals the slope of the chord PQ (i.e., where the tangent to the curve is parallel to the secant line).

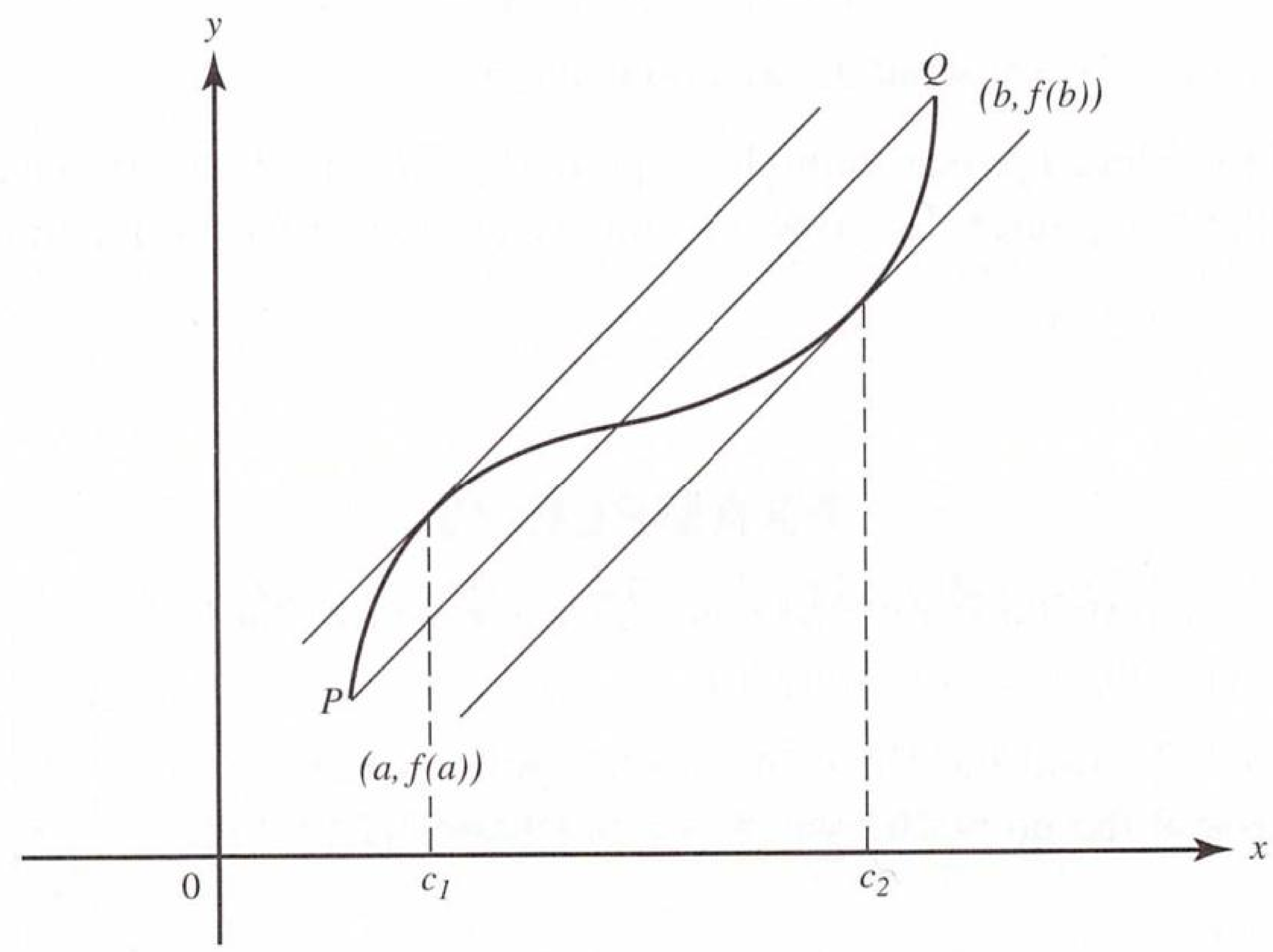

FIGURE N3–9

We will often refer to the Mean Value Theorem by its initials, MVT.

Rolle's Theorem

If, in addition to the hypotheses of the MVT, it is given that $f(a) = f(b) = k$, then there is a number, c, between a and b such that $f'(c) = 0$. This special case of the MVT is called Rolle's Theorem, as seen in Figure N3–10 for $k = 0$.

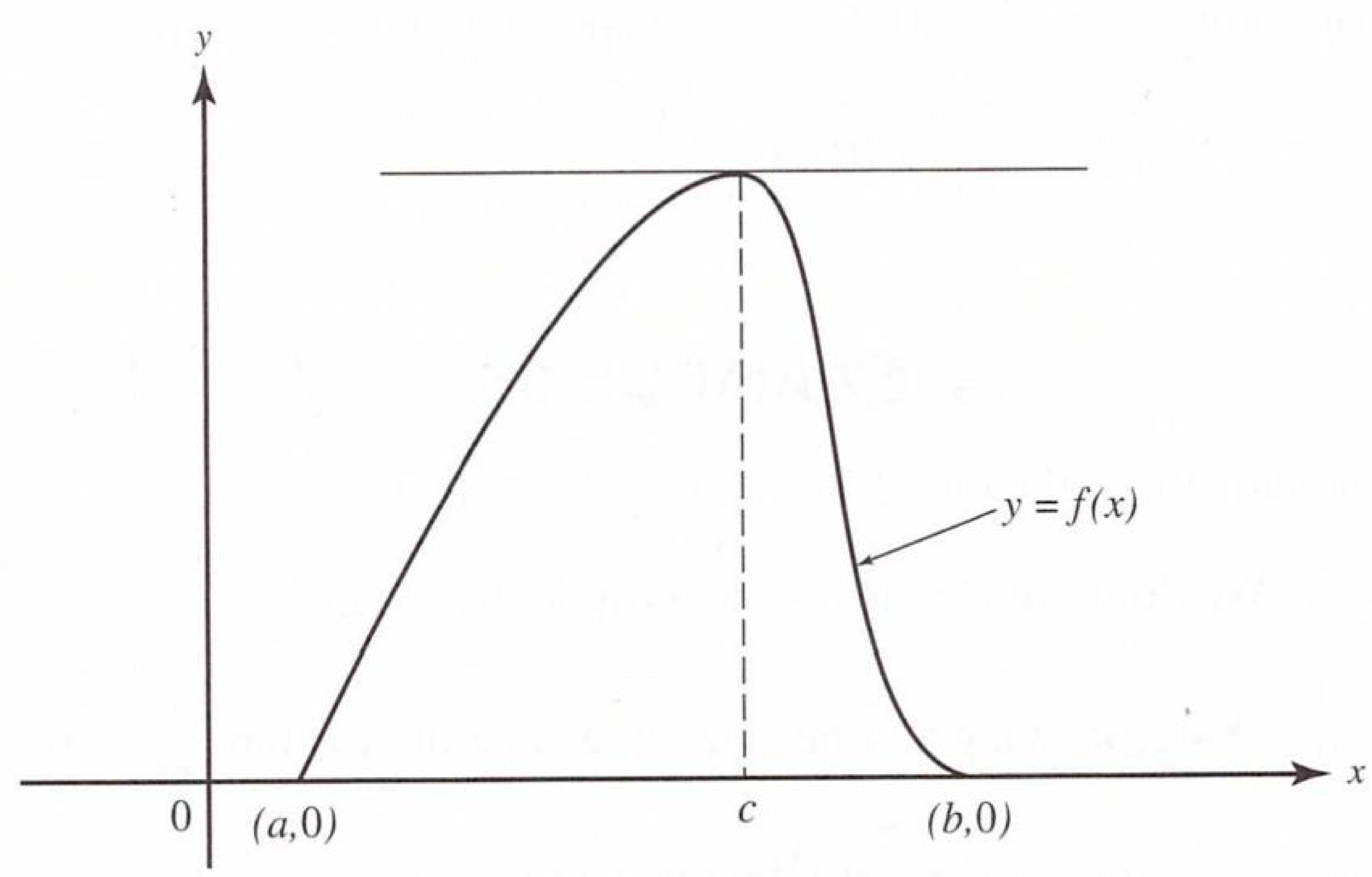

FIGURE N3–10

The Mean Value Theorem is one of the most useful laws when properly applied.

EXAMPLE 36

You left home one morning and drove to a cousin's house 300 miles away, arriving 6 hours later. What does the Mean Value Theorem say about your speed along the way?

SOLUTION: Your journey was continuous, with an average speed (the average rate of change of distance traveled) given by

$$\frac{\Delta\text{distance}}{\Delta\text{time}} = \frac{300 \text{ miles}}{6 \text{ hours}} = 50 \text{ mph}.$$

Furthermore, the derivative (your instantaneous speed) existed everywhere along your trip. The MVT, then, guarantees that at least at one point your instantaneous speed was equal to your average speed for the entire 6-hour interval. Hence, your car's speedometer must have read exactly 50 mph at least once on your way to your cousin's house.

EXAMPLE 37

Demonstrate Rolle's Theorem using $f(x) = x \sin x$ on the interval $[0,\pi]$.

SOLUTION: First, we check that the conditions of Rolle's Theorem are met:

(1) $f(x) = x \sin x$ is continuous on $[0,\pi]$ and exists for all x in $[0,\pi]$.
(2) $f'(x) = x \cos x + \sin x$ exists for all x in $[0,\pi]$.
(3) $f(0) = 0 \sin 0 = 0$ and $f(\pi) = \pi \sin \pi = 0$.

Hence there must be a point, $x = c$, in the interval $[0,\pi]$ where $f'(c) = 0$. Using the calculator to solve $x \cos x + \sin x = 0$, we find $c = 2.029$ (to three decimal places). As predicted by Rolle's Theorem, $0 \le c \le \pi$.
Note that this result indicates that at $x = c$ the line tangent to f is horizontal. The MVT (here as Rolle's Theorem) tells us that any function that is continuous and differentiable must have at least one turning point between any two roots.

J.* INDETERMINATE FORMS AND L'HÔPITAL'S RULE

BC ONLY

Limits of the following forms are called *indeterminate*:

$$\frac{0}{0} \text{ or } \frac{\infty}{\infty},\ 0 \cdot \infty,\ \infty - \infty,\ 0^0,\ 1^\infty,\ \infty^0$$

To find the limit of an indeterminate form of the type $\frac{0}{0}$ or $\frac{\infty}{\infty}$, we apply L'Hôpital's Rule, which involves taking derivatives of the functions in the numerator and denominator. In the following, a is a finite number. The rule has several parts:

(a) If $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} g(x) = 0$ and if $\lim\limits_{x\to a} \frac{f'(x)}{g'(x)}$ exists†, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)};$$

if $\lim\limits_{x\to a} \frac{f'(x)}{g'(x)}$ does not exist, then L'Hôpital's Rule cannot be applied.

*Although this a *required* topic only for BC students, AB students will find L'Hôpital's Rule very helpful.

†The limit can be finite or infinite ($+\infty$ or $-\infty$).

BC ONLY

(b) If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \infty$, the same consequences follow as in case (a). The rules in (a) and (b) both hold for one-sided limits.

(c) If $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = 0$ and if $\lim_{x\to\infty} \frac{f'(x)}{g'(x)}$ exists, then

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)};$$

if $\lim_{x\to\infty} \frac{f'(x)}{g'(x)}$ does not exist, then L'Hôpital's Rule cannot be applied. (Here the notation "$x \to \infty$" represents either "$x \to +\infty$" or "$x \to -\infty$.")

(d) If $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$, the same consequences follow as in case (c).

In applying any of the above rules, if we obtain $\frac{0}{0}$ or $\frac{\infty}{\infty}$ again, we can apply the rule once more, repeating the process until the form we obtain is no longer indeterminate.

EXAMPLE 38

$\lim_{x\to 3} \frac{x^2-9}{x-3}$ is of type $\frac{0}{0}$ and thus equals $\lim_{x\to 3} \frac{2x}{1} = 6$.
(Compare with Example 12, pages 94 and 95.)

EXAMPLE 39

$\lim_{x\to 0} \frac{\tan x}{x}$ is of type $\frac{0}{0}$ and therefore equals $\lim_{x\to 0} \frac{\sec^2 x}{1} = 1$.

EXAMPLE 40

$\lim_{x\to -2} \frac{x^3+8}{x^2-4}$ (Example 13, page 95) is of type $\frac{0}{0}$ and thus equals $\lim_{x\to -2} \frac{3x^2}{2x} = -3$, as before. Note that $\lim_{x\to -2} \frac{3x^2}{2x}$ is *not* the limit of an indeterminate form!

EXAMPLE 41

$\lim_{h\to 0} \frac{e^h-1}{h}$ is of type $\frac{0}{0}$ and therefore equals $\lim_{h\to 0} \frac{e^h}{1} = 1$.

EXAMPLE 42

$\lim_{x\to\infty} \frac{x^3-4x^2+7}{3-6x-2x^3}$ (Example 20, page 96) is of type $\frac{\infty}{\infty}$, so that it equals

$\lim_{x\to\infty} \frac{3x^2-8x}{-6-6x^2}$, which is again of type $\frac{\infty}{\infty}$. Apply L'Hôpital's Rule twice more:

$$\lim_{x\to\infty} \frac{6x-8}{-12x} = \lim_{x\to\infty} \frac{6}{-12} = -\frac{1}{2}.$$

For this problem, it is easier and faster to apply the Rational Function Theorem!

BC ONLY

EXAMPLE 43

Find $\lim_{x\to\infty} \frac{\ln x}{x}$.

SOLUTION: $\lim_{x\to\infty} \frac{\ln x}{x}$ is of type $\frac{\infty}{\infty}$ and equals $\lim_{x\to\infty} \frac{1/x}{1} = 0$.

EXAMPLE 44

Find $\lim_{x\to 2} \frac{x^3+8}{x^2+4}$.

SOLUTION: $\lim_{x\to 2} \frac{x^3+8}{x^2+4} = \frac{16}{8} = 2$.

BEWARE: L'Hôpital's Rule applies only to indeterminate forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$. Trying to use it in other situations leads to incorrect results, like this:

$$\lim_{x\to 2} \frac{x^3+8}{x^2+4} = \lim_{x\to 2} \frac{3x^2}{2x} = 3 \quad \text{(WRONG!)}$$

L'Hôpital's Rule can be applied also to indeterminate forms of the types $0 \cdot \infty$ and $\infty - \infty$, if the forms can be transformed to either $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

EXAMPLE 45

Find $\lim_{x\to\infty} x \sin \frac{1}{x}$.

SOLUTION: $\lim_{x\to\infty} x \sin \frac{1}{x}$ is of the type $\infty \cdot 0$. Since $x \sin \frac{1}{x} = \frac{\sin 1/x}{1/x}$ and, as $x \to \infty$, the latter is the indeterminate form $\frac{0}{0}$, we see that

$$\lim_{x\to\infty} x \sin \frac{1}{x} = \lim_{x\to\infty} \frac{-\frac{1}{x^2}\cos\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to\infty} \cos\frac{1}{x} = 1.$$

(Note the easier solution $\lim_{x\to\infty} x \sin \frac{1}{x} = \lim_{x\to\infty} \frac{\sin \frac{1}{x}}{\frac{1}{x}} = 1$.)

Other indeterminate forms, such as 0^0, 1^∞, and ∞^0, may be resolved by taking the natural logarithm and then applying L'Hôpital's Rule.

BC ONLY

EXAMPLE 46

Find $\lim_{x\to 0}(1+x)^{1/x}$.

SOLUTION: $\lim_{x\to 0}(1+x)^{1/x}$ is of type 1^{∞}. Let $y = (1+x)^{1/x}$, so that

$\ln y = \frac{1}{x}\ln(1+x)$. Then $\lim_{x\to 0}\ln y = \lim_{x\to 0}\frac{\ln(1+x)}{x}$, which is of type $\frac{0}{0}$. Thus,

$$\lim_{x\to 0}\ln y = \lim_{x\to 0}\frac{\frac{1}{1+x}}{1} = \frac{1}{1} = 1,$$

and since $\lim_{x\to 0}\ln y = 1$, $\lim_{x\to 0} y = e^1 = e$.

EXAMPLE 47

Find $\lim_{x\to\infty} x^{1/x}$.

SOLUTION: $\lim_{x\to\infty} x^{1/x}$ is of type ∞^0. Let $y = x^{1/x}$, so that $\ln y = \frac{1}{x}\ln x = \frac{\ln x}{x}$ (which, as $x \to \infty$, is of type $\frac{\infty}{\infty}$). Then $\lim_{x\to\infty}\ln y = \frac{1/x}{1} = 0$, and $\lim_{x\to\infty} y = e^0 = 1$.

For more practice, redo the Practice Exercises on pages 102–107, applying L'Hôpital's Rule wherever possible.

K. RECOGNIZING A GIVEN LIMIT AS A DERIVATIVE

It is often extremely useful to evaluate a limit by recognizing that it is merely an expression for the definition of the derivative of a specific function (often at a specific point). The relevant definition is the limit of the difference quotient:

$$f'(c) = \lim_{h\to 0}\frac{f(c+h)-f(c)}{h}.$$

EXAMPLE 48

Find $\lim_{h\to 0}\frac{(2+h)^4-2^4}{h}$.

SOLUTION: $\lim_{h\to 0}\frac{(2+h)^4-2^4}{h}$ is the derivative of $f(x) = x^4$ at the point where $x = 2$. Since $f'(x) = 4x^3$, the value of the given limit is $f'(2) = 4(2^3) = 32$.

EXAMPLE 49

Find $\lim_{h\to 0}\frac{\sqrt{9+h}-3}{h}$.

SOLUTION: $\lim_{h\to 0}\frac{\sqrt{9+h}-3}{h} = f'(9)$, where $f(x) = \sqrt{x}$. The value of the limit is $\frac{1}{2}x^{-1/2}$ when $x = 9$, or $\frac{1}{6}$.

EXAMPLE 50

Find $\lim\limits_{h \to 0} \frac{1}{h}\left(\frac{1}{2+h} - \frac{1}{2}\right)$.

SOLUTION: $\lim\limits_{h \to 0} \frac{1}{h}\left(\frac{1}{2+h} - \frac{1}{2}\right) = f'(2)$, where $f(x) = \frac{1}{x}$.

Verify that $f'(2) = -\frac{1}{4}$ and compare with Example 17, page 95.

EXAMPLE 51

Find $\lim\limits_{h \to 0} \frac{e^h - 1}{h}$.

SOLUTION: $\lim\limits_{h \to 0} \frac{e^h - 1}{h} = f'(0)$, where $f(x) = e^x$. The limit has value e^0 or 1 (see also Example 41, on page 130).

EXAMPLE 52

Find $\lim\limits_{x \to 0} \frac{\sin x}{x}$.

SOLUTION: $\lim\limits_{x \to 0} \frac{\sin x}{x}$ is $f'(0)$, where $f(x) = \sin x$, because we can write

$$f'(0) = \lim_{x \to 0} \frac{\sin(0 + x) - \sin 0}{x} = \lim_{x \to 0} \frac{\sin x}{x}.$$

The answer is 1, since $f'(x) = \cos x$ and $f'(0) = \cos 0 = 1$. Of course, we already know that the given limit is the basic trigonometric limit with value 1. Also, L'Hôpital's Rule yields 1 as the answer immediately.